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A DISCRETE APPROACH FOR ANALYSIS OF SOUND TRANSMISSION IN PIPES COUPLED WITH COMPACT COMMUNICATING DEVICES

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A new discrete method is presented for acoustic modelling of pipes that are coupled by compact devices. This is based on an approximate source analogy that allows the communicating devices to be modelled as continuous two-port elements. One of the areas in which this method finds application is the acoustic analysis of mufflers with multiple perforated pipes. For this case, the present formulation assumes fundamental mode propagation for simplicity. Also presented in the paper are applications in which perforate holes are modelled as continuous visco-thermal pipes, the results of which are compared with the lumped parameter modelling of perforate impedance. In general, the present method is computationally simpler and more versatile than the more commonly known distributed parameter method.

1. INTRODUCTION

An area in which the theory of sound transmission in coupled cavities finds application is the acoustic analysis of mufflers with multiple perforated pipes, which are extensively used in engine exhaust systems to reduce the tailpipe noise. In this context, the so-called distributed parameter method has received considerable attention by authors aiming to improve its efficacy in practical applications [1]. In the distributed parameter method one assumes plane wave propagation in the pipes and models the perforations as swamped around the pipe wall continuously, the effect of which is accounted for by using a lumped parameter impedance model, usually an empirical or a semi-empirical one.

A disadvantage of the distributed parameter method is the lack of flexibility in incorporating acoustic models of the communicating perforations in the analysis. In particular, the concept of lumped perforate impedance is not completely satisfactory for the modelling of a coupling device, or a distribution of them, that may be used in place of the usual holes. A discrete approach may then be more suitable and it is the purpose of this paper to introduce such an approach.

The idea of modelling a perforate discretely is not new. Sullivan [2] proposed a method in which perforations are lumped into a number of quasi-static control volumes and their effect is accounted for by using a lumped parameter model of perforate impedance. This method is now known as the segmentation method. Kergomard *et al.* [3] presented, for the case of two cavities communicating via a single hole, a consistent lumped parameter model for the impedance of a hole and used this in a periodic system formulation to study the transmission of plane sound waves in the two-pipe configuration.

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The new contributions of the present analysis are: (1) a perforate hole is generalized to the concept of a compact acoustic device that may have a two-port acoustic model and, (2) a method is developed for integrating such a device into the analysis of sound wave transmission in a number of parallel pipes that communicate with each other via that device. The method was initially developed for the analysis of sound transmission in certain pipes that communicate via few non-uniform pipes. Later, it proved to be computationally simpler and more versatile than the distributed parameter method in the analysis of mufflers with multiple perforated pipes.

No attempt is made in this paper to study the effects of different communicating device designs. The possibilities are many and can be investigated by the proposed method. Two applications are presented, however, for the validation of the analysis. These are the straight-through and the cross-flow (or, plug) resonators that have been studied experimentally and theoretically by Sullivan [2]. In these resonators the communicating devices are in the form of circular holes and it is an objective of the present applications to compare the modelling of a perforate hole as a continuous tube and as lumped impedance.

2. THEORETICAL FORMULATION

2.1. BASIC CONSIDERATIONS AND ASSUMPTIONS

Consider a number of parallel uniform hard-walled pipes of finite length enclosed in a hard-walled pipe that acts as casing. The sound fields in these pipes are assumed to be coupled by branch-like devices inserted on the walls of the inner pipes. The communicating devices may be in the form of a simple circular hole, a louvre, a pipe, or a miniature chamber or muffler. The problem is to derive a transfer matrix describing the transmission of sound waves from the input to the output side of the pack. The solution of this problem requires a knowledge of the transfer matrices describing transmission of sound waves in the pipes across a discontinuity created by a coupling device, and the transmission of sound waves along the device itself. The latter is assumed to be available as an acoustic two-port, and the former is deduced approximately from one-dimensional analysis of sound transmission in the pipe, in which the inlet, or outlet, aperture of a communicating device is assumed to be compact enough to be modelled as a simple source. The part of the coupling device that protrudes into the pipe is assumed to be small compared to the wavelengths in question and have no flow–acoustic interaction. Then, assuming a uniform axial mean flow and isentropic plane sound waves with $\exp(-\mathrm{i}\omega t)$ time dependence, where ω is the radian frequency, i is the unit imaginary number and t denotes the time; the momentum and continuity equations for a uniform pipe with a communicating device centered at axial co-ordinate $x = \xi$, where x denotes the pipe axis, can be expressed as, respectively,

$$\frac{\partial p}{\partial x} + \rho c\left(M\frac{\partial}{\partial x} - \mathrm{i}k\right)v = 0 \quad \left(M\frac{\partial}{\partial x} - \mathrm{i}k\right)p + \rho c\frac{\partial v}{\partial x} = 2m\delta(x-\xi). \tag{1}$$

Here, p is the acoustic pressure, v is the partical velocity, $k = \omega/c$ is the wavenumber, c is the speed of sound, ρ is the ambient density, M is the Mach number of the mean flow velocity, $\delta(x)$ denotes a Dirac function at $x = 0$, and m is given by $m = \rho cQ/2S$, where S is the pipe cross-sectional area and Q is the rate of volume injection at the aperture into per unit volume of the pipe. Equations (1) are the momentum and continuity equations of reference [4] written for the case of uniform mean flow and discrete mass injection. The assumption of uniform mean flow is plausible when there is no mean flow through the communicating device. With through flow, the axial mean flow varies slightly in the vicinity of the device, but it is known that the effect of mean flow velocity gradient can be taken into account accurately by assuming an axially averaged uniform mean flow velocity [1, 4]. Thus, it is understood that, equations (1) are to be implemented by using an axially averaged uniform mean flow velocity if through flow is present. It should also be noted that, in a solid section of a pipe having no communicating devices, the continuity equation holds without the source term. In this case, the solution of equations (1) is classical and is conveniently expressed in terms of the pressure wave components, p^+ and p^-, where $p = p^+ + p^-$ and $\rho cv = p^+ - p^-$, as $p^+(x) = p^+(0)\exp(\mathrm{i}kx/(1+M))$ and $p^-(x) = p^-(0)\exp(-\mathrm{i}kx/(1-M))$, which correspond to waves travelling in $+x$ and $-x$ directions respectively.

A relationship between the pressure wave components just upstream, $x = \xi_-$, and just downstream, $x = \xi_+$, the discontinuity created by a communicating device can be obtained from the solution of equations (1). Presented in Appendix A is a solution which embodies a heuristic correction, ε, to ξ_- and ξ_+ as $\xi_- = \xi - \varepsilon$ and $\xi_+ = \xi + \varepsilon$, respectively. In the present analysis this correction, which aims to account for the effects of evanescent waves that may be created in the close vicinity of a communicating device, is assumed to be acoustically compact, that is, $k\varepsilon \ll 1$. Then, the relationship between the pressure wave components upstream and downstream of a communicating device is given simply as

$$\begin{bmatrix} p^+(\xi_+) \\ p^-(\xi_+) \end{bmatrix} = \begin{bmatrix} p^+(\xi_-) \\ p^-(\xi_-) \end{bmatrix} + \frac{\rho c Q}{2}\begin{bmatrix} (1+\bar{M})^{-1} \\ -(1-\bar{M})^{-1} \end{bmatrix}, \tag{2}$$

where $\bar{M}$ denotes the average of the upstream and downstream mean flow Mach numbers.

Acoustically, a communicating device can be conceived as entraining a lumped inertia or sound waves. Both cases are of interest here and will be considered separately. In the latter case, the source strength, Q, is given by $Q = \int U\,\mathrm{d}A$ where A is the cross-sectional area and U denotes the normal component of the particle velocity at the inlet or outlet of a communicating device, which is assumed to be a uniform pipe terminal. Then, including viscous and thermal losses and a uniform mean flow but restricting propagation to the fundamental mode, U can be expressed as

$$\rho c U = h^+ P^+ + h^- P^-, \tag{3}$$

where P^+ and P^- denote the acoustic pressure wave components in positive and negative flow directions, respectively, and the coefficients $h^\pm$ are given in Appendix B. By convention, flow is positive in inlet to outlet direction of the communicating device. For isentropic sound propagation, $h^\pm = \pm 1$.

The pressure wave components at the inlet and outlet of a communicating device are related by the two-port relationship

$$\begin{bmatrix} P^+ \\ P^- \end{bmatrix}_{outlet} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_{device} \begin{bmatrix} P^+ \\ P^- \end{bmatrix}_{inlet}. \tag{4}$$

Here, the device transfer matrix is assumed to be available from a separate analysis. For example, for a uniform narrow tube, $T_{11} = \exp(\mathrm{i}kK^+ l)$, $T_{22} = \exp(\mathrm{i}kK^- l)$, $T_{12} = T_{21} = 0$, where the propagation constants $K^\pm$ are given in Appendix B, and l denotes the length of the tube, which may include a two-sided end-correction for the added-mass effect. In the case of isentropic propagation, the propagation constants are given by $K^\pm = \pm 1/(1 \pm M)$, where M denotes the mean flow velocity Mach number in the communicating tube.

If a communicating passage between the pipes is relatively short, as for a hole drilled in a thin-walled pipe, this may be assumed to entrain a lumped vibrating mass in the manner of the classical Helmholtz resonator model. Under this approximation, communicating pipes are coupled through the equation of motion of this mass, which is expressed, traditionally, as an impedance relationship. A lumped impedance model of a communicating device can be invoked by using equation (3) with $h^{\pm} = \pm 1$ and replacing equation (4) by

$$\begin{bmatrix} P^+ \\ P^- \end{bmatrix}_{outlet} = \begin{bmatrix} 1-\zeta/2 & \zeta/2 \\ -\zeta/2 & 1+\zeta/2 \end{bmatrix}_{device} \begin{bmatrix} P^+ \\ P^- \end{bmatrix}_{inlet}, \tag{5}$$

where ζ is the normalized impedance of the communicating device.

This formulation can be closed now by the condition of pressure continuity at the inlet, or outlet, aperture of a communicating device, that is,

$$p^+(\xi_-) + p^-(\xi_-) = P^+ + P^-. \tag{6}$$

This is an approximate 1-D implementation of the continuity of pressure in the physical system. Clearly, the correction ε has to be compact, as assumed in equation (2), if this condition is to be physically meaningful. Since communicating devices are assumed to be compact, this limits the largest value of 2ε to about the characteristic size of a device. The actual value must be determined heuristically. For example, one may use a value that provides a good fit to the test results, or inspect the sensitivity of the results to different corrections. For a perforated pipe, the total correction, 2ε, is applied by reducing the length of solid pipe sections by this amount and is subsequently called the pitch correction.

2.2. MODELLING OF COUPLED MULTIPLE PARALLEL PIPES

In this section, the foregoing formulation is applied to derive a wave transfer relationship across a pack of multiple parallel perforated pipes that communicate with each other through an arbitrary number of identical devices that are distributed axially on the surface of the inner pipes. Pipes are numbered from 1 to n, pipe 1 being the enclosing pipe. It suffices to derive the wave transfer relationship for a pack in which the inner pipes all have a single communicating device at the same axial position. The configuration is shown in Figure 1. Once the transfer matrix for this case is determined, it can be used in cascade with solid pipe elements to compute the wave transfer across a pack of multiple parallel pipes having any axial distribution of communicating devices. If one of the inner pipes does not have

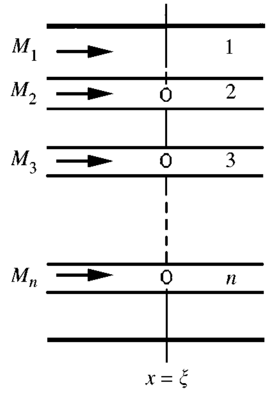

Figure 1. Multiple pipes having communicating devices at same axial position.

a communicating device at a common axial location, this can be accounted for simply by setting the inlet and outlet cross-sectional areas of the device equal to zero, see equations (17)–(20).

Referring to Figure 1, for pipe j, equation (2) is now written as

$$\mathbf{P}_j(\xi_+) = \mathbf{P}_j(\xi_-) + m_j\mathbf{B}_j, \quad j = 1, 2, \ldots, n, \tag{7}$$

where

$$|m_j| = |\rho_j c_j Q_j / 2S_j|, \tag{8}$$

$$\mathbf{P}_j(x) = \begin{bmatrix} p_j^+(x) \\ p_j^-(x) \end{bmatrix}, \qquad \mathbf{B}_j = \begin{bmatrix} (1 + M_j)^{-1} \\ -(1 - M_j)^{-1} \end{bmatrix}. \tag{9}$$

Here S_j denotes the cross-sectional area of pipe j and the subscript j refers to pipe j. If there is mean through flow, the mean flow Mach number M_j should be understood as the average of its upstream and downstream values. From equations (3) and (8), it follows that

$$m_j = -\frac{A_{inlet,j}\mathbf{h}_j\mathbf{P}_{inlet,j}}{2S_j}, \quad j = 2, 3, \ldots, n, \tag{10}$$

$$m_1 = \sum_{j=2}^{n} \frac{A_{outlet,j}\mathbf{h}_j\mathbf{P}_{outlet,j}}{2S_1}, \tag{11}$$

where A denotes a cross-sectional area of the communicating device, the subscripts "inlet" and "outlet" refer to the inlet of the communicating device in an inner pipe ($j = 2, 3, \ldots, n$) and its outlet into the enclosing pipe, respectively, and,

$$\mathbf{h}_j = [h_j^+ \;\; h_j^-], \quad \mathbf{P}_{inlet,j} = \begin{bmatrix} P_{inlet,j}^+ \\ P_{inlet,j}^- \end{bmatrix}, \quad \mathbf{P}_{outlet,j} = \begin{bmatrix} P_{outlet,j}^+ \\ P_{outlet,j}^- \end{bmatrix}.$$

The acoustic two-port describing the sound wave transfer between the inlet and outlet of a communicating device mounted on the wall of pipe j is now expressed as

$$\mathbf{P}_{outlet,j} = \mathbf{T}_j\mathbf{P}_{inlet,j}, \quad j = 2, 3, \ldots, n, \tag{12}$$

where $\mathbf{T}_j$ denotes the (2×2) transfer matrix in equation (4) or equation (5). Applying equation (6) to the parallel pipes,

$$\mathbf{EP}_{inlet,j} = \mathbf{EP}_j(\xi_-), \qquad \mathbf{EP}_{outlet,j} = \mathbf{EP}_1(\xi_-), \quad j = 2, 3, \ldots, n, \tag{13}$$

where $\mathbf{E} = [1 \;\; 1]$. Using equation (12), the foregoing equations are combined as

$$\mathbf{P}_{inlet,j} = \mathbf{A}_{1,j}\mathbf{P}_1(\xi_-) + \mathbf{A}_{2,j}\mathbf{P}_j(\xi_-), \quad j = 2, 3, \ldots, n, \tag{14}$$

where

$$\mathbf{A} = [\mathbf{A}_{1,j} \;\; \mathbf{A}_{2,j}] = \begin{bmatrix} \mathbf{ET}_j \\ \mathbf{E} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}. \tag{15}$$

Hence, upon substituting equation (14) into equations (10) and (11), and the result into equation (7), the relationship between the pressure wave components upstream and downstream of a bank of communicating devices is obtained as

$$\begin{bmatrix} \mathbf{P}_1(\xi_+) \\ \mathbf{P}_2(\xi_+) \\ \mathbf{P}_3(\xi_+) \\ \vdots \\ \mathbf{P}_n(\xi_+) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} & \cdots & \mathbf{T}_{1n} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{T}_{31} & \mathbf{0} & \mathbf{T}_{33} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{T}_{n1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{P}_1(\xi_-) \\ \mathbf{P}_2(\xi_-) \\ \mathbf{P}_3(\xi_-) \\ \vdots \\ \mathbf{P}_n(\xi_-) \end{bmatrix}. \tag{16}$$

Here,

$$\mathbf{T}_{11} = \mathbf{I} + \sum_{j=2}^{n} \frac{A_{outlet,j}\mathbf{B}_1\mathbf{h}_j\mathbf{T}_j\mathbf{A}_{1,j}}{2S_1}, \tag{17}$$

$$\mathbf{T}_{1j} = \frac{A_{outlet,j}\mathbf{B}_1\mathbf{h}_j\mathbf{T}_j\mathbf{A}_{2,j}}{2S_1}, \tag{18}$$

$$\mathbf{T}_{jj} = \mathbf{I} - \frac{A_{inlet,j}\mathbf{B}_j\mathbf{h}_j\mathbf{A}_{2,j}}{2S_j}, \tag{19}$$

$$\mathbf{T}_{j1} = -\frac{A_{inlet,j}\mathbf{B}_j\mathbf{h}_j\mathbf{A}_{1,j}}{2S_j}, \tag{20}$$

where $j = 2, 3, \ldots, n$ and $\mathbf{I}$ denotes a (2×2) unit matrix. Equation (16) is derived for n pipes all having a communicating device at the same axial position, but it is now clear that, if one of the pipes does not have a communicating device at this position, the corresponding form of equation (16) is obtained simply by inputting the inlet and outlet cross-sectional areas of that device as zero. This ability to deal with any permutation of coupling devices at the same axial position in a unified manner provides considerable simplification in the modelling of multiple perforated pipes having axially staggered perforate patches.

2.3. THE TWO-PIPE ELEMENT

A computer algorithm is available for the implementation of the foregoing theory for any number of parallel pipes and communicating device models. For the purpose of the present paper, however, it suffices to consider applications of the simplest configuration, that is, the two-pipe case and it is expedient to give equation (16) for this case explicitly. It is

$$\begin{bmatrix} \mathbf{P}_1(\xi_+) \\ \mathbf{P}_2(\xi_+) \end{bmatrix} = \begin{bmatrix} \mathbf{I} + a_{11}\mathbf{M}_1 & a_{12}\mathbf{M}_1 \\ a_{21}\mathbf{M}_2 & \mathbf{I} + a_{22}\mathbf{M}_2 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1(\xi_-) \\ \mathbf{P}_2(\xi_-) \end{bmatrix}. \tag{21}$$

Here,

$$\mathbf{M}_j = \begin{bmatrix} (1 + M_j)^{-1} & (1 + M_j)^{-1} \\ -(1 - M_j)^{-1} & -(1 - M_j)^{-1} \end{bmatrix}, \quad j = 1, 2, \tag{22}$$

$$a_{11} = \frac{A_{outlet,2}[h_2^+(T_{2,11} - T_{2,21}) - h_2^-(T_{2,22} - T_{2,12})]}{2S_1(T_{2,11} + T_{2,12} - T_{2,22} - T_{2,21})}, \tag{23}$$

$$a_{22} = \frac{A_{inlet,2}[h_2^+(T_{2,22} + T_{2,21}) - h_2^-(T_{2,11} + T_{2,12})]}{2S_2(T_{2,11} + T_{2,12} - T_{2,22} - T_{2,21})}, \tag{24}$$

$$a_{12} = \frac{A_{outlet,2}(T_{2,11}T_{2,22} - T_{2,12}T_{2,21})(-h_2^+ + h_2^-)}{2S_1(T_{2,11} + T_{2,12} - T_{2,22} - T_{2,21})}, \tag{25}$$

$$a_{21} = \frac{A_{inlet,2}(-h_2^+ + h_2^-)}{2S_2(T_{2,11} + T_{2,12} - T_{2,22} - T_{2,21})}, \tag{26}$$

where the elements of matrix $\mathbf{T}_2$ are denoted as

$$\mathbf{T}_2 = \begin{bmatrix} T_{2,11} & T_{2,12} \\ T_{2,21} & T_{2,22} \end{bmatrix}. \tag{27}$$

3. NUMERICAL RESULTS

Two important applications of the two-pipe element are the straight-through and the cross-flow resonators. Because of their relative simplicity, these resonators provide a convenient setting for a discussion of the proposed method. Experimental results obtained under carefully controlled conditions are available for these resonators [2]. These are compared in this section with the predictions of the present theory.

A perforated pipe is modelled as a pipe having axial and radial discrete distributions of communicating devices, which are, for the resonators considered, circular holes drilled in the pipe wall. A radial distribution of identical holes at a given axial location is called a hole-bank. Hole-banks are separated by solid pipe sections. It is assumed that the interaction of hole-banks through evanescent waves is negligible. Since the fundamental mode propagation is assumed, the method is not sensitive to the number of holes in a hole-bank. Therefore, insofar as the determination of the source strengths is concerned, a hole-bank can be considered as a single communicating hole of area equal to the sum of the cross-sectional areas of the holes in the hole-bank. However, the parameters of a lumped impedance model, or a continuous tube model, of a hole must be determined by using the actual hole diameter.

The first step in the formulation of sound wave transmission characteristics of a resonator employing a perforated pipe is to derive a four-port model for the perforated section by combining equation (21) in cascade with the transfer matrices of the solid pipe sections that separate the hole-banks. This four-port model describes the relationship between the pressure wave components at the left and right ends of a perforated section in the resonator as

$$\begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix}_{left} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix}_{right}, \tag{28}$$

where the subscripts "1" and "2" refer to the enclosing and perforated pipes respectively. Calculation of this (4×4) matrix requires only matrix multiplication. The transfer matrix of a given resonator is then obtained by application of the boundary conditions. This step is shown in the following separately for the two resonators considered. Boundary conditions are assumed to be given in terms of the reflection coefficient R, which is defined as the quotient $R = p^-/p^+$.

3.1. STRAIGHT-THROUGH RESONATOR

In this case, as shown in Figure 2, the reflection coefficients of the end-caps and, therefore, $R_{1,left}$ and $R_{1,right}$ are assumed to be known and the wave transfer is required as a relationship between $\mathbf{P}_{2,left}$ and $\mathbf{P}_{2,right}$. Multiplying the first of equations (28) by $\mathbf{Y}_{1,left} = [R_{1,left} \quad -1]$ from the left gives

$$\mathbf{Y}_{1,left}\mathbf{S}_{11}\mathbf{R}_{1,right}p^{+}_{1,right} + \mathbf{Y}_{1,left}\mathbf{S}_{12}\mathbf{P}_{2,right} = 0, \tag{29}$$

where $\mathbf{R}_{1,right} = \{1 \quad R_{1,right}\}$. Solving this for $p^{+}_{1,right}$ and inserting the result in the equation corresponding to the second row of equation (28) yields the desired relationship:

$$\mathbf{P}_{2,left} = \left[\mathbf{S}_{22} - \frac{\mathbf{S}_{21}\mathbf{R}_{1,right}\mathbf{Y}_{1,left}\mathbf{S}_{12}}{\mathbf{Y}_{1,left}\mathbf{S}_{11}\mathbf{R}_{1,right}}\right]\mathbf{P}_{2,right}. \tag{30}$$

A well-studied straight-through resonator is that of Sullivan [2]. The inner pipe of this resonator has a uniformly drilled 2·49 mm diameter hole pattern of porosity 0·037. The perforated patch is $L = 66{\cdot}7$ mm long and is positioned centrally in the resonator with its ends at $L_R = L_L = 6{\cdot}4$ mm from the outer pipe end-caps. The inside diameters of the inner and outer pipes are $D_2 = 49{\cdot}3$ mm and $D_1 = 101{\cdot}6$ mm, respectively, and the thickness of the inner pipe is 0·81 mm. The details of the hole pattern and the test temperature are not given in reference [2]. In the present calculations, temperature is taken as 20°C and the perforate patch is assumed to consist of equally spaced 7 hole-banks each having 12 holes. Figure 3 shows the transmission loss computed by using the present theory with and without visco-thermal losses in the holes taken into account. In computing these results, the ambient fluid is assumed to be dry air, and a pitch correction of $0{\cdot}7d$ and a two-sided hole end-correction of $0{\cdot}75d$ are used, where d is the hole diameter. The characteristic corresponding to the modelling of the holes as narrow pipes agrees fairly well with the experimental results of reference [2] except in the vicinity of the main peak, where they are few dB higher. The present end-correction is the same as the corresponding term in the lumped perforate impedance model of Sullivan [2], who used a test-based constant value for the resistive part of the model. The pitch correction was required in order for the sharp spike to occur at about the experimentally observed frequency. The transmission loss of this resonator was computed also by the present method, with the same pitch correction, and the distributed parameter method, using the lumped perforate impedance model of reference [2] in both cases. The former solution is shown in Figure 3. The latter solution, which is not shown, is similar to this characteristic, but the sharp spike occurs at about 2400 Hz, which is appreciably lower than the test value. The axial pitch of the perforations

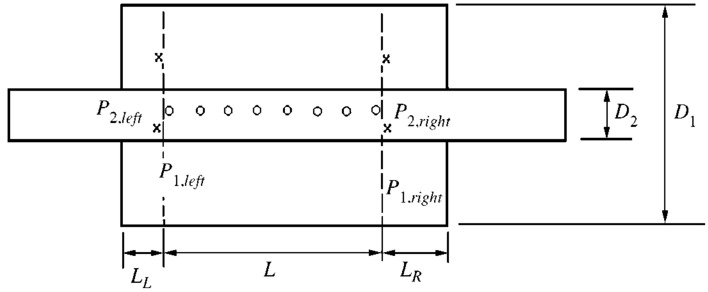

Figure 2. A straight-through resonator.

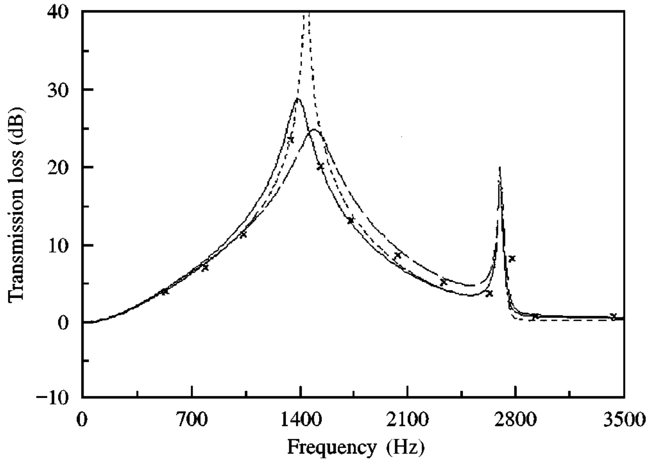

Figure 3. Transmission loss of a straight-through resonator with zero mean flow; ——, the present theory with continuous visco-thermal pipe model for holes; - - - -, the present theory with continuous isentropic pipe model for holes; — —, the present theory with lumped impedance model [2] for holes; × Experiment [2].

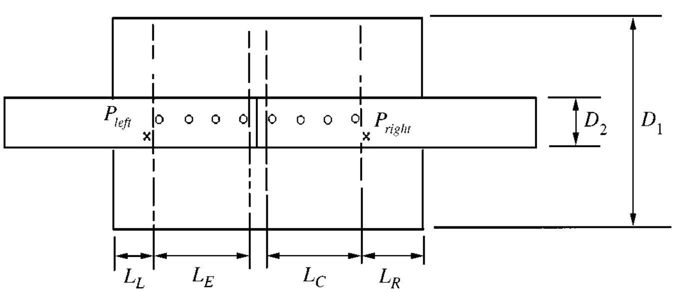

Figure 4. A cross-flow resonator.

controls the frequency of this spike. The distributed parameter method can be sensitive to this parameter to the extent to which it is represented in the lumped impedance model of perforate impedance.

3.2. CROSS-FLOW RESONATOR

In this resonator, shown in Figure 4, the inner pipe has a plug between two perforate patches so that, when there is mean flow, this is forced through the holes. In this case, the reflection coefficients of the end-caps of the outer pipe and of the two sides of the plug are assumed to be known and the wave transfer relationship is required as a relationship between $\mathbf{P}_{left}$ and $\mathbf{P}_{right}$. This is best obtained by splitting the resonator into three components in cascade, namely, a plugged expansion, an annular pipe and a plugged contraction. The wave transfer relation across the plugged expansion and contraction elements can be derived now as in the straight-through resonator case.

For the plugged expansion (Figure 5(a)), the reflection coefficients $R_{1,left}$ and $R_{2,right}$ are known and the wave transfer is required as a relationship between $\mathbf{P}_{2,left}$ and $\mathbf{P}_{1,right}$. Multiplying the first of equations (28) by $\mathbf{Y}_{1,left} = [R_{1,left} \;\; -1]$ from the left gives

$$\mathbf{Y}_{1,left}\mathbf{S}_{11}\mathbf{P}_{1,right} + \mathbf{Y}_{1,left}\mathbf{S}_{12}\mathbf{R}_{2,right}p^{+}_{2,right} = 0, \tag{31}$$

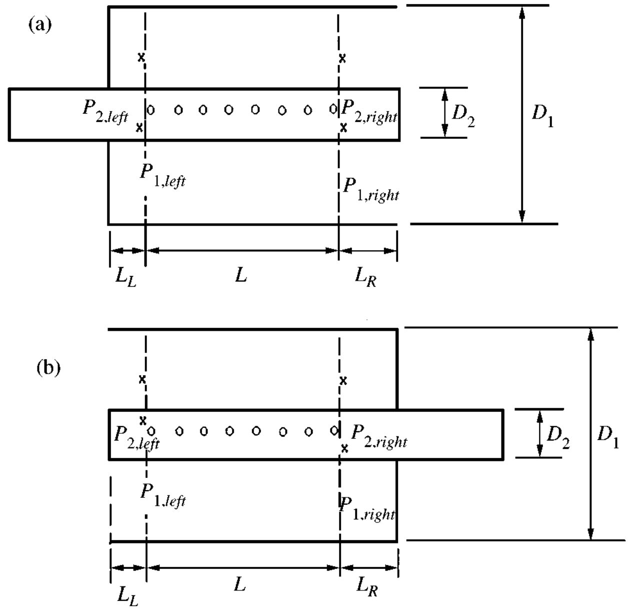

Figure 5. Components of a cross-flow resonator, (a) plugged expansion, (b) plugged contraction.

where $\mathbf{R}_{2,right} = \{1 \;\; R_{2,right}\}$. Solving this for $p^+_{2,right}$ and inserting the result into the equation corresponding to the second row of equation (28) yields

$$\mathbf{P}_{2,left} = \left[\mathbf{S}_{21} - \frac{\mathbf{S}_{22}\mathbf{R}_{2,right}\mathbf{Y}_{1,left}\mathbf{S}_{11}}{\mathbf{Y}_{1,left}\mathbf{S}_{12}\mathbf{R}_{2,right}}\right]\mathbf{P}_{1,right}. \tag{32}$$

For the plugged contraction, Figure 5(b), the reflection coefficients $R_{2,left}$ and $R_{1,right}$ are known and the wave transfer is required as a relationship between $\mathbf{P}_{1,left}$ and $\mathbf{P}_{2,right}$. Obviously, this is symmetrical to the plugged expansion and, therefore, the wave transfer relationship can be obtained from equation (32) simply by interchanging the subscripts 1 and 2:

$$\mathbf{P}_{1,left} = \left[\mathbf{S}_{12} - \frac{\mathbf{S}_{11}\mathbf{R}_{1,right}\mathbf{Y}_{2,left}\mathbf{S}_{22}}{\mathbf{Y}_{2,left}\mathbf{S}_{21}\mathbf{R}_{1,right}}\right]\mathbf{P}_{2,right}. \tag{33}$$

Hence, the wave transfer relationship across the resonator in Figure 4 can be expressed as

$$\mathbf{P}_{left} = \left[\mathbf{S}_{21} - \frac{\mathbf{S}_{22}\mathbf{R}_{2,right}\mathbf{Y}_{1,left}\mathbf{S}_{11}}{\mathbf{Y}_{1,left}\mathbf{S}_{12}\mathbf{R}_{2,right}}\right]_{exp}\mathbf{T}_{pipe}\left[\mathbf{S}_{12} - \frac{\mathbf{S}_{11}\mathbf{R}_{1,right}\mathbf{Y}_{2,left}\mathbf{S}_{22}}{\mathbf{Y}_{2,left}\mathbf{S}_{21}\mathbf{R}_{1,right}}\right]_{con}\mathbf{P}_{right}. \tag{34}$$

where $\mathbf{T}_{pipe}$ denotes the transfer matrix of the annular pipe connecting the plugged expansion and contraction units. For isentropic plane sound wave propagation, this is of the form described in section 2.1.

Sullivan has presented an experimental and theoretical study of a cross-flow muffler with and without mean flow [2]. The inner pipe of the cross-flow resonator considered by Sullivan has a uniformly drilled 2·49 mm diameter hole pattern of porosity 0·039. The perforated patches in the plugged expansion and contraction modules are each $L_E = L_C = 128{\cdot}6$ mm long and are flush with the plug and end-caps, i.e., $L_R = L_L = 0$. The

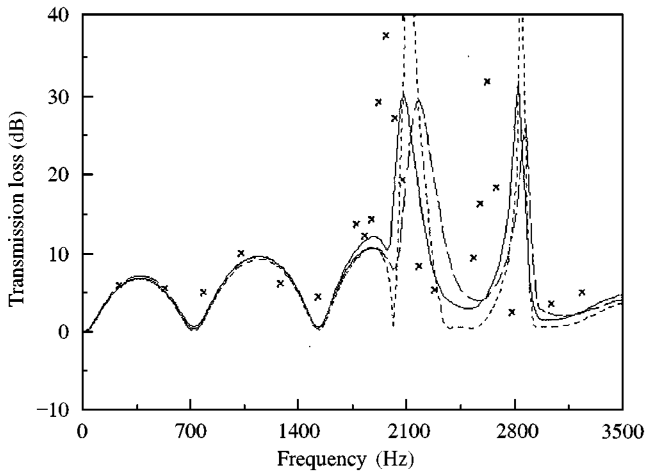

Figure 6. Transmission loss of a cross-flow resonator with zero mean flow; ——, the present theory with continuous visco-thermal pipe model for holes; - - - -, the present theory with continuous isentropic pipe model for holes; — —, the present theory with lumped impedance model [2] for holes; × Experiment [2].

inside diameters of the inner and outer pipes are $D_2 = 49{\cdot}3$ mm and $D_1 = 101{\cdot}6$ mm, respectively, and the thickness of the inner pipe is 0·81 mm. The details of the hole pattern and, in the zero mean flow case, the test temperature are not given in reference [2]. In the present calculations, the latter is assumed to be 20°C. The perforate patch is assumed to consist of equally spaced 13 hole-banks, each having 12 holes in both the plugged expansion and contraction modules. Figure 6 shows the transmission loss computed for the case of zero mean flow by using the present theory with and without visco-thermal losses in the holes taken into account. In computing these results, the ambient fluid is assumed to be dry air, and a two-sided hole end-correction of 0·75 d is used, where d is the hole diameter. No pitch correction was applied, as it did not improve the correlation with the measured results [2], some of which are shown in Figure 6. The present end-correction is the same as the corresponding term in the lumped perforate impedance model used by Sullivan. The characteristic corresponding to the modelling of the holes as narrow pipes agrees fairly well with the theoretical results of reference [2] (not shown here). Also shown in Figure 6 is the transmission loss computed by using the lumped impedance model of Sullivan [2], again with zero pitch correction.

These characteristics change considerably with mean flow, but the foregoing models fail to predict these changes. This is due to the fact that a convective continuous narrow pipe model of a perforate hole cannot model by itself the effects of through flow on sound transmission. Therefore, it is necessary to incorporate these effects into a continuous tube model of a perforate hole. A simpler approach is to use a lumped parameter model of hole impedance that allows for the effects of through flow. This approach is feasible here, because the perforate thickness is small. Figure 7 shows the transmission loss of the cross-flow resonator computed for an inlet mean flow velocity Mach number of 0·05, by using the present theory with the lumped perforate impedance model that was developed by Sullivan [2] for the through flow case. Other parameters are the same as those described for the zero mean flow case, except the ambient temperature, which is 74°C for this case. Mean flow in the plugged contraction module is assumed to increase linearly from zero to full flow in the inner pipe and decrease linearly from full flow to zero in the chamber, and similarly but in the opposite sense in the plugged expansion module. Also shown in Figure 7 are some

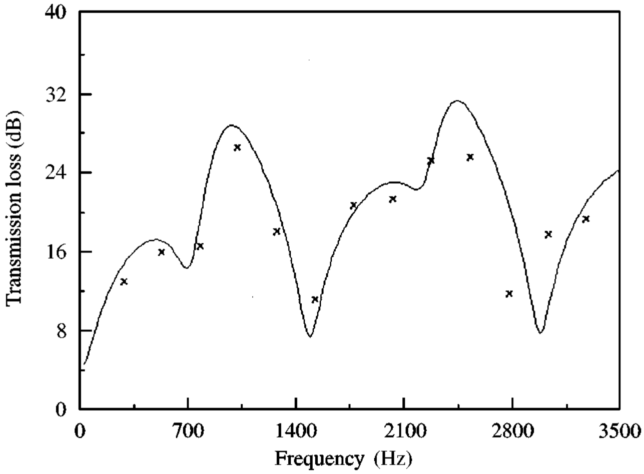

Figure 7. Transmission loss of a cross-flow resonator with mean flow, $M = 0{\cdot}05$; ——, the present theory with lumped impedance model [2] for holes; × Experiment [2].

experimental results of reference [2]. The present results are also in agreement with the results of the segmentation method [2] (not shown here).

4. CONCLUSION

A new method has been presented for acoustic modelling of pipes that are coupled by compact communicating devices. Muffler components incorporating perforated pipes were emphasized as a significant application area of the method; however, its potential is not limited with these applications.

The implementation of the present formulation will be limited by the compactness condition of the communicating devices and the condition of fundamental mode propagation in the pipes. It is possible to remove these restrictions under the same conceptual framework, but at the cost of added mathematical complexity. In this context, such an extension of the present method has yielded promising initial results in the modelling of the three-dimensional effects in the casing enclosing the perforated pipes. This extension, which is of importance for relatively short casings, will be published at a later date.

In general, the present method is computationally simpler and more versatile than the distributed parameter method in acoustical analysis of multiple perforated pipe mufflers. It has been applied to a variety of automotive mufflers including multiple-path ones such as the three-pass muffler. In relatively high porosity cases, it gives almost the same results as the distributed parameter method in the frequency ranges of interest in automotive applications when used with the same lumped parameter model of perforate impedance. In the relatively low porosity cases, however, the present method should be preferred to the distributed parameter method because of its sensitivity to the axial pitch of perforations. Also, the present method is more versatile in dealing with cases in which perforate patches in different pipes are staggered axially so that they overlap partially. The computational advantage of this method comes from the fact that no complex eigenvalue problem or root extraction problem needs to be solved as in the distributed parameter method.

A hole perhaps provides the most practical way of coupling two pipes, but this does not preclude the possibility of developing other coupling devices that may find technological use. The present method allows the use of two-port models of communicating devices and can be used to study the effects of such devices at the design stage.

The main problem in the modelling of a communicating device as an acoustic two port, is accounting for the effects of mean through flow and further research is needed in this area. For a communicating device such as a hole drilled on a thin-walled pipe, this problem can be resolved by using a lumped parameter model.

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APPENDIX A: ACOUSTIC WAVE TRANSFER ACROSS A SIMPLE SOURCE PLANE

This appendix presents a solution of equation (1). Using matrix notation, equation (1) can be expressed as

$$\partial/\partial x\, \mathbf{P}(x) = \mathbf{H}\mathbf{P}(x) + m\mathbf{B}\delta(x-\xi), \tag{A.1}$$

where

$$\mathbf{P}(x) = \begin{bmatrix} p^+(x) \\ p^-(x) \end{bmatrix}, \quad \mathbf{H} = \mathrm{i}k\begin{bmatrix} (1+M)^{-1} & 0 \\ 0 & -(1-M)^{-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} (1+M)^{-1} \\ -(1-M)^{-1} \end{bmatrix}. \tag{A.2}$$

The general solution of equation (A.1) is [5]

$$\mathbf{P}(x) = [\mathbf{T}]_0^x \mathbf{c} + m\int_0^x \mathbf{K}(x,\kappa)\mathbf{B}(\kappa)\delta(\kappa-\xi)\,\mathrm{d}\kappa. \tag{A.3}$$

Here, **c** denotes a vector of integration constants and the Green function $\mathbf{K}(x,\kappa)$ is

$$\mathbf{K}(x,\kappa) = [\mathbf{T}]_0^x([\mathbf{T}]_0^\kappa)^{-1}, \tag{A.4}$$

where

$$[\mathbf{T}]_0^x = [\mathbf{T}]_{\xi+\varepsilon}^x[\mathbf{T}]_{\xi-\varepsilon}^{\xi+\varepsilon}[\mathbf{T}]_0^{\xi-\varepsilon}, \tag{A.5}$$

$$[\mathbf{T}]_0^{\xi-\varepsilon} = \exp(\mathbf{H}_- x), \tag{A.6}$$

$$[\mathbf{T}]_{\xi+\varepsilon}^x = \exp(\mathbf{H}_+(x-\xi-\varepsilon)), \tag{A.7}$$

where $\mathbf{H}_+$ and $\mathbf{H}_-$ denote the matrix $\mathbf{H}$ evaluated for $M = M_+$ an $M = M_-$, where M_+ and M_- denote the mean flow Mach numbers for $x \leqslant \xi - \varepsilon$ and $x \geqslant \xi + \varepsilon$, respectively, and ε denotes a heuristic correction that is introduced to account for the effects of evanescent waves that may be created in the vicinity of the source discontinuity. It is assumed that the mean flow Mach number in the interval $\xi - \varepsilon \leqslant x \leqslant \xi + \varepsilon$ is uniform and equal to the mean value $\bar{M} = (M_+ + M_-)/2$. Then,

$$[\mathbf{T}]_{\xi-\varepsilon}^{\xi+\varepsilon} = \exp(2\bar{\mathbf{H}}\varepsilon), \tag{A.8}$$

where $\bar{\mathbf{H}}$ denotes the matrix $\mathbf{H}$ evaluated for $M = \bar{M}$. The Green function can be expressed as

$$\mathbf{K}(x, \kappa) = \exp(\mathbf{H}_-(x - \kappa)), \quad x \leqslant \xi - \varepsilon, \tag{A.9}$$

$$\mathbf{K}(x, \kappa) = \exp(\mathbf{H}_+(x - \kappa)), \quad x \geqslant \xi + \varepsilon, \tag{A.10}$$

$$\mathbf{K}(x, \kappa) = \exp(\bar{\mathbf{H}}(x - \kappa)), \quad \xi - \varepsilon \leqslant x \leqslant \xi - \varepsilon. \tag{A.11}$$

Hence, restricting x to the positive axis, equation (A.3) becomes

$$\mathbf{P}(x) = [\mathbf{T}]_0^x\mathbf{c} + m\exp(\bar{\mathbf{H}}(x - \xi)\bar{\mathbf{B}}H(x - \xi)), \tag{A.12}$$

where $\mathrm{H}(x)$ denotes a Heaviside function at $x = 0$ and $\bar{\mathbf{B}}$ denotes the matrix $\mathbf{B}$ evaluated for $M = \bar{M}$. The vector $\mathbf{c}$ is determined by writing the foregoing equation for $x = \xi - \varepsilon$.

$$\mathbf{c} = ([\mathbf{T}]_0^{\xi-\varepsilon})^{-1}\mathbf{P}(\xi - \varepsilon) = \exp(-\mathbf{H}_-(\xi - \varepsilon))\mathbf{P}(\xi - \varepsilon). \tag{A.13}$$

Then, from equation (A.10),

$$[\mathbf{T}]_0^x\mathbf{c} = \exp(\mathbf{H}_+(x - \xi - \varepsilon))\exp(2\mathbf{H}\varepsilon))\mathbf{P}(\xi - \varepsilon), \quad x \geqslant \xi + \varepsilon. \tag{A.14}$$

Hence, putting $x = \xi + \varepsilon$ into equation (A.12) yields the following wave transfer relationship across a simple source discontinuity:

$$\exp(-\bar{\mathbf{H}}\varepsilon)\mathbf{P}(\xi + \varepsilon) = \exp(\bar{\mathbf{H}}\varepsilon)\mathbf{P}(\xi - \varepsilon) + m\bar{\mathbf{B}}. \tag{A.15}$$

To the author's knowledge, this result has not been published elsewhere. If ε is compact, i.e. $k\varepsilon \ll 1$, then equation (A.15) simplifies to

$$\mathbf{P}(\xi_+) = \mathbf{P}(\xi_-) + m\bar{\mathbf{B}}. \tag{A.16}$$

This relationship is used in the present analysis as the basic relationship describing the sound wave transfer across a communicating device. If there is no through flow across the communicating device, then $\bar{M} = M$ and $\bar{\mathbf{B}} = \mathbf{B}$.

APPENDIX B: SOUND TRANSMISSION IN A NARROW PIPE

Summarized in this appendix are expressions for sound wave transfer in a uniform narrow pipe carrying a mean flow. The sound pressure is given by [6]

$$p(x) = p^+(x) + p^-(x), \tag{B.1}$$

where

$$p^{\pm}(x) = p^{\pm}(0)\exp(\mathrm{i}kK^{\pm}x). \tag{B.2}$$

For a circular pipe, the propagation constants $K^{\pm}$ are computed from the dispersion equation

$$\left[\frac{K}{1-MK}\right]^2 = -\left[\frac{\mathrm{J}_0(\beta a)}{\mathrm{J}_2(\beta a)}\right]\left[\gamma + (\gamma-1)\frac{\mathrm{J}_2(\sigma\beta a)}{\mathrm{J}_0(\sigma\beta a)}\right], \quad \beta a = s\sqrt{\mathrm{i}(1-KM)}. \tag{B.3}$$

Here, a denotes the radius of the pipe, J_n denotes a Bessel function of order n, γ is the ratio of specific heat coefficients, σ^2 denotes the Prandtl number and s is the shear wavenumber, which is defined as $s = a\sqrt{(\rho\omega/\mu)}$, where μ is the shear viscosity coefficient and ω is the radian frequency, and $\exp(-\mathrm{i}\omega t)$ time dependence is assumed for the fluctuating quantities.

The particle velocity is given by

$$\rho c v(x, r) = h^+(r)p^+(x) + h^-(r)p^-(x), \tag{B.4}$$

where r denotes the radial co-ordinate and

$$h^{\pm}(r) = [K^{\pm}/1 - K^{\pm}]\,[1 - (\mathrm{J}_0(\beta^{\pm}r)/\mathrm{J}_0(\beta^{\pm}a))]. \tag{B.5}$$

Integration of equation (B.4), which is equation (3) of the main text, over the pipe cross-section then yields

$$h^{\pm} = [-K^{\pm}/(1-K^{\pm})]\,[\mathrm{J}_2(\beta^{\pm}a)/\mathrm{J}_0(\beta^{\pm}a)]. \tag{B.6}$$

This average value occurs in equations (10) and (11).